1. *Let (x_n) be sequence of real numbers.*
(a) Show that (x_n) converges imply (x_n) is bounded
(b) Let $x \in \mathbb{R}$. Show that $x_n \to x$ iff $\liminf x_n = x = \limsup x_n$.

Solution: (a) Let $x_n \to x$ then for a $\epsilon > 0$ there is a M such that $x - \epsilon < x_n < x + \epsilon \quad \forall \quad n > M$. Set $L = \max_{1\leq k\leq M} |x_k|$. So we get $|x_n| \leq \max\{L,\ |x-\epsilon|,\ |x+\epsilon|\} \quad \forall\ n \geq 1$.

(b) We have $\liminf x_n = \sup_n y_n = \lim_{n\to\infty} y_n, \quad y_n = \inf_{k\geq n} x_k$ and $y_n \leq y_{n+1}$. Similarly $\limsup x_n = \inf_n z_n = \lim_{n\to\infty} z_n, \quad z_n = \sup_{k\geq n} x_k$ and $z_{n+1} \leq z_n$. So we have $y_n \leq x_n \leq z_n$. If $\liminf x_n = x = \limsup x_n$ i.e $\lim y_n = \lim z_n = x$ then by sandwich theorem we have $\lim x_n = x$. Now assume $x_n \to x$ then for each $\epsilon > 0$ there exist M such that $x - \epsilon < x_n < x + \epsilon \quad \forall\ n > M$. Then from the definition of y_n and z_n we have $x - \epsilon < y_n \leq z_n < x + \epsilon \quad \forall\ n > M$. So we get $\liminf x_n = \lim y_n = x = \lim z_n = \limsup x_n$. □

2. *(a) Say true or false: every bounded sequence (x_n) has a subsequence (x_{n_k}) such that $x_{n_k} \to \limsup x_n \in \mathbb{R}$.*
(b) Prove that every bounded sequence has a convergent subsequence.
(c) Suppose (a_n) and (b_n) are bounded sequences in $\mathbb{R}$. Then prove that $(a_n + b_n)$ is a bounded sequence and $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$.

Solution (a) True. Since $\limsup x_n$ is the supremum of all limits point of (x_n).
(b) Bolzano-Weierstrass theorem.
(c) Since (a_n) and (b_n) are bounded we have $|a_n| \leq M_1$ and $|b_n| \leq M_2 \quad \forall\ n$. Then we have $|a_n + b_n| \leq |a_n| + |b_n| \leq M_1 + M_2 \ \ \forall\ n$. So $(a_n + b_n)$ is bounded.
We have $a_j + b_j \leq \sup_{k\geq n} a_n + \sup_{k\geq n} b_n \ \ \forall\ j \geq n \Rightarrow \sup_{k\geq n}(a_n + b_n) \leq \sup_{k\geq n} a_n + \sup_{k\geq n} b_n$. Now we get $\lim_{n\to\infty} \sup_{k\geq n}(a_n + b_n) \leq \lim_{n\to\infty} \sup_{k\geq n} a_n + \lim_{n\to\infty} \sup_{k\geq n} b_n$. Which will imply $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$. □

3. *(a) If $|x| < 1$ and $q \in \mathbb{R}$ show that $\sum n^q x^n$ converges.*
(b) Prove that $n^{\frac{1}{n}} \to 1$ and use it to show that $\sum a_n$ is converges where $a_n = n^{\frac{1}{n}} - 1$.

Solution (a) We have $\lim\limits_{n\to\infty} |n^q x^n|^{\frac{1}{n}} = |x| < 1$ here we use that $n^{\frac{1}{n}} \to 1$. Then by root test we have the convergence of $\sum n^q x^n$.
(b) We have $n^{\frac{1}{n}} \geq 1, \quad n \geq 2$. Let $x_n = n^{\frac{1}{n}} - 1 \geq 0$ then $(1 + x_n)^n = n$. Now binomial expansion will give $n = 1 + nx_n + \frac{n(n-1)}{2}x_n^2 + \cdots + x_n^n \Rightarrow 1 + \frac{n(n-1)}{2}x_n^2 < n$ so we get $\frac{n(n-1)}{2}x_n^2 < n - 1 < n$ $\Rightarrow x_n < \sqrt{\frac{1}{n-1}}$. So we get $\lim x_n = 0 \ \ i.e \ \ n^{\frac{1}{n}} \to 1$.
Now $\lim |a_n|^{\frac{1}{n}} = \lim(n^{\frac{1}{n}} - 1) = 0 < 1$, so by root test we get the convergence of $\sum a_n$. □

4. *Let $\sum a_n$ converges absolutely. Then show that $\sum a_n$ converges and every rearrangement $\sum a_{k_n}$ also converges to $\sum a_n$. Show also that any rearrangement $\sum a_{k_n}$ converges absolutely and $\sum |a_n| = \sum |a_{k_n}|$.*

Solution See the Theorem 3.5F of Methods of Real Analysis by Richard R. Goldberg. □

5. *(a) Let A be a non-empty subset of real line $\mathbb{R}$ and define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = \inf_{a\in A} |x - a|$, $x \in \mathbb{R}$. Show that f is continuous on $\mathbb{R}$.*
(b) Let f be a continuous function on $\mathbb{R}$ and $x \in \mathbb{R}$. Suppose $f(x) \neq 0$, Show that there are $\delta,\ \eta > 0$ such that $|f(r)| > \eta$ for all $r \in (x - \delta, x + \delta)$.
(c) Let f and g be a continuous function on $\mathbb{R}$ and $x \in \mathbb{R}$. Suppose $f(x) \neq g(x)$ show that there is a $\delta > 0$ such that $f(r) \neq g(r)$ for all $r \in (x - \delta, x + \delta)$.

Solution (a) We have $|x - a| \leq |x - y| + |y - a| \ \ x,\ y \in \mathbb{R},\ a \in A$. We get $|x-a| \leq |x-y|+|y-a| \leq |x-y|+|y-a|$ i.e $\inf_{a\in A} |x-a| \leq |x-y|+|y-a| \Rightarrow |y-a| \geq f(x)-|x-y|$ whcih will give $f(y) \geq f(x)-|x-y|$ i.e $f(x)-f(y) \leq |x-y|$, similary we can get $f(y)-f(x) \leq |x-y|$. Now we have $|f(x) - f(y)| \leq |x - y|$ which will give the continuity of f.
(b) Let assume $f(x) > 0$, since f is continuous at x then for each $\epsilon > 0$ there exist a $\delta > 0$ such that $|f(r) - f(x)| < \epsilon \ \ \forall\ r \in (x - \delta, x + \delta)$. Chose $\epsilon = \frac{f(x)}{2}$ then we have $f(r) > f(x) - \frac{f(x)}{2} = \frac{f(x)}{2}$ for all $r \in (x - \delta, x + \delta)$. So if take $\eta = \frac{f(x)}{2}$ we are done. If $f(x) < 0$ the do the same thing as above with the function $h := -f$.
(c) Define $h := f - g$ with $h(x) \neq 0$ then by above problem we get there are η and δ such that $|h(r)| > \eta > 0$ for all $r \in (x - \delta, x + \delta)$, i.e $f(r) \neq g(r)$ (i.e $h(r) \neq 0$) for all $r \in (x - \delta, x + \delta)$. □

6. *(a) If $f : [0, 1] \longrightarrow [0, 1]$ is a continuous function. Then prove that there is $x \in [0, 1]$ such that $f(x) = x$.*
(b)If $f : [0, 1] \longrightarrow [0, 2]$ is a continuous function. Then prove that there is $x \in [0, 1]$ such that $f(x) = 2 - 2x$

Solution (a) If $f(0) = 0$ of $f(1) = 1$ then we are done. Let assume $f(0) > 0$ and $f(1) < 1$. Now consider $g(x) = f(x)-x$ then $g(0) > 0$ and $g(1) = f(1)-1 < 0$ then by intermediate value theorem there is a x in $[0, 1]$ such that $g(x) = 0$ i.e $f(x) = x$.
(b) If $f(0) = 2$ of $f(1) = 0$ then we are done. Le assume $f(0) < 2$ and $f(1) > 0$. Let $g(x) = f(x) - 2 + 2x$ then $g(0) = f(0) - 2 < 0$ and $g(1) = f(1) > 0$, now intermediate value theorem will give the result. □

7. *Let $f : I \longrightarrow \mathbb{R}$ be an uniformly continuous function and (x_n) be a sequence in I.*
(a) If (x_n) is cauchy show that $(f(x_n))$ is a cauchy.
(b) If (x_n) and (y_n) are sequence in I such that $x_n \to x$, $y_n \to x$ for some $x \in \mathbb{R}$ then show that $\lim f(x_n) = \lim f(y_n)$.

Solution (a) Since $\{x_n\}_n$ is a cauchy sequence, we have for each $\delta > 0$ there is $M \in \mathbb{N}$ (M depends on δ) such that $|x_n - x_m| < \delta,\ \ \forall\ n, m > M$. Now uniform continuity of f will give

$$|f(x_n) - f(x_m)| < \epsilon \ \ \forall\ n, m > M.$$

So $\{f(x_n)\}_n$ is a cauchy sequence.
(b) Since $x_n \to x$, $y_n \to x$ for some $x \in \mathbb{R}$ so both (x_n) and (y_n) both are cauchy therefore $(f(x_n))$ and $(f(y_n))$ are also cauchy. So both limit $\lim f(x_n)$ and $\lim f(y_n)$ exist.
Now for any $\delta > 0$ we can find $M \in \mathbb{N}$ such that $|x_n - x| < \frac{\delta}{2}$ *and* $|y_n - x| < \frac{\delta}{2}\ \forall\ n \geq M$. Now uniform continuity of f give , for any $\epsilon > 0$ we have for $n > M$

$$|f(x_n) - f(y_n)| < \epsilon \ \ as \ \ |x_n - y_n| \leq |x_n - x| + |x - y_n| < \delta.$$

The above will give $\lim f(x_n) = \lim f(y_n)$. □